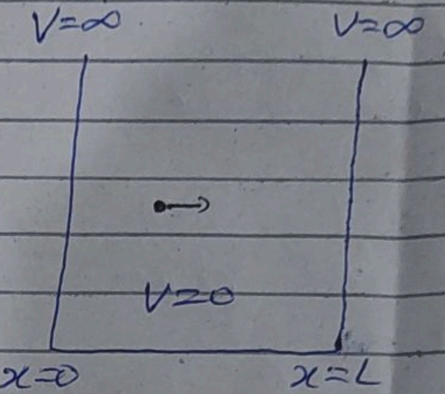
Schrodinger wave equation for a particle in one-dimentional Box

Consider a particle that is constrained to move only in x direction from $x=0$ to $x=L$



Schrodinger wave eq in operator form

$$\hat{H}\psi = E\psi \longrightarrow (1)$$

($\hat{H}$ = hamiltonian operator in x-direction)

$$\hat{H} = \frac{-h^2}{8\pi^2 m}\frac{d^2}{dx^2} + V$$

Putting value in eq (1)

$$\frac{-h^2}{8\pi^2 m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$$\frac{-h^2}{8\pi^3 m}\frac{d^2\psi}{dx^2} + V\psi - E\psi = 0$$

$$\frac{h^2}{8\pi^2 m}\frac{d^2\psi}{dx^2}(E-V)\psi = 0$$

Multiplying throughout by $\frac{8\pi^2 m}{h^2}$

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0}$$

Outside the box $V = \infty$ $\quad x<0$ or $x>L$

$$\frac{d^2\psi}{d^2x}\ \frac{8\pi^2 m}{h^2}(E-\infty)\psi = 0$$

This is possible ~~due~~ only when $\psi = 0$
i.e particle is not outside the box

$$\frac{d^2\psi}{d^2x} + (-\infty)\psi = 0$$

$$\psi = \frac{1}{\infty}\frac{d^2\psi}{dx^2} = 0$$

Inside the Box $V = 0$ $\quad 0 < x$ or $x < L$

$$\frac{d^2\psi}{d^2x} + \frac{8\pi^2 m}{h^2}(E-0)\psi = 0$$

$$\frac{d^2\psi}{d^2x} + \frac{8\pi^2 m}{h^2}E\psi = 0$$ 2nd ~~+~~

$\rightarrow K^2$ $\qquad K^2 = \frac{2mE}{\hbar^2}$

$$\frac{d^2\psi}{d^2x} + K^2\psi = 0$$ → 2nd order differential equation

Wave function can be represented in the term of the trignometric functions

$\psi(x) = A\sin Kx + B\cos Kx$ $\qquad$ (A, B = arbitrary constant

when $x = 0$

$$\psi_{(0)} = 0$$

$$0 = A\sin 0x + B\cos 0x$$ $\qquad$ ($\sin 0 = 0$, $\cos 0 = 1$

$$0 = 0 + B$$

$$B = 0$$

When $x = l$

$$\psi_{(l)} = 0$$

$$0 = A\sin kl + B\cos kl \qquad \because B = 0$$

$$0 = A\sin kl + 0$$

$$A\sin kl = 0 \qquad \because kl = n\pi$$

$$\sin kl = 0 = \sin n\pi$$

By comparing

$$kl = n\pi$$

$$k = \frac{n\pi}{l} \text{ or } k^2 = \frac{n^2\pi^2}{l^2} \qquad \because k^2 = \frac{2mE}{\hbar^2}$$

Now

$$\frac{n^2\pi^2}{l^2} = \frac{2mE}{\hbar^2}$$

$$E = \frac{\hbar^2 n^2 \pi^2}{2ml^2} \qquad \because \hbar = \frac{h}{2\pi}$$

$$\hbar^2 = \frac{h^2}{4\pi^2}$$

$$E = \frac{h^2 n^2 \cancel{\pi^2}}{4\cancel{\pi^2}\, 2ml^2}$$

$$\boxed{E = \frac{h^2 n^2}{8ml^2}} \qquad \because n = 1x, 4x, 9x, 16x \ldots$$

So energy of a particle in one dimensional is quantised

When $n=1$ $\qquad$ $n=2$

$$E_1 = \frac{\hbar^2}{8mL^2} \qquad E_2 = \frac{4\hbar^2}{}$$

$n=3$ $\qquad$ $n=4$

$$E_3 = \frac{9h^2}{8mL^2} \qquad E_4 = \frac{16h^2}{8mL^2}$$

The probability of finding the particle in a small space between x and $x+dx$ is given by

$$\psi^2_{(x)}\, dx$$

$$\int_0^L \psi^2_{(x)}\, dx = 1 \qquad \because \psi = A\sin\left(\frac{n\pi x}{L}\right)$$

$$A^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx = 0$$

$$A = \sqrt{\frac{2}{L}}$$

$$\boxed{\psi_{(x)} = \frac{2}{L} \sin\frac{n\pi}{L}x}$$